Optimum Design Concepts

- Methods used for design optimization does not depend on the field of engineering.
- Broad classification of the optimization

  Optimality criteria methods (indirect methods)

  Search methods (direct search methods)
- Optimality criteria are the conditions a function must satisfy at its minimum point.
- Study of optimality conditions are necessary regardless of the method of optimization used.

- Gradient Vector (the vector of first derivatives):

For a function of n variables $f(x_1, x_2, \ldots, x_n)$

$$c_i = \frac{\partial f(\boldsymbol{x})}{\partial x_i}$$

$$\boldsymbol{c} = \nabla f(\boldsymbol{x}) = \frac{\partial f}{\partial x} = grad f = \begin{Bmatrix} \frac{\partial f(\boldsymbol{x})}{\partial x_1} \\ \frac{\partial f(\boldsymbol{x})}{\partial x_2} \\ \ldots \\ \frac{\partial f(\boldsymbol{x})}{\partial x_n} \end{Bmatrix}$$

- Geometrically, the gradient vector at point $\boldsymbol{x}^p$ is normal to the tangent plane to the function at that point, and points in the direction of maximum increase in the function.

- Hessian Matrix (the matrix of second derivatives):

For a function of n variables
$f(x_1, x_2, \ldots, x_n)$

$$H_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) \qquad \boldsymbol{H} = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} f(\mathbf{x}) & \frac{\partial^2}{\partial x_1 \partial x_2} f(\mathbf{x}) & \ldots & \frac{\partial^2}{\partial x_1 \partial x_n} f(\mathbf{x}) \\ \frac{\partial^2}{\partial x_2 \partial x_1} f(\mathbf{x}) & \frac{\partial^2}{\partial x_2^2} f(\mathbf{x}) & \ldots & \frac{\partial^2}{\partial x_2 \partial x_n} f(\mathbf{x}) \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial^2}{\partial x_n \partial x_1} f(\mathbf{x}) & \frac{\partial^2}{\partial x_n \partial x_2} f(\mathbf{x}) & \ldots & \frac{\partial^2}{\partial x_n^2} f(\mathbf{x}) \end{bmatrix}$$

- Hessian matrix is always a symmetric matrix

$$\frac{\partial^2}{\partial x_i \partial x_j} f = \frac{\partial^2}{\partial x_j \partial x_i} f$$

- Taylor Series Expansion:
  - for a function with one variable, Taylor series expansion about x^p

$$f(x) = f(x^p) + \frac{df(x^p)}{dx}(x - x^p) + \frac{1}{2}\frac{d^2 f(x^p)}{dx^2}(x - x^p)^2 + R$$

At a small distance d from x^p

$x = x^p + d$

$$f(x^p + d) = f(x^p) + \frac{df(x^p)}{dx}d + \frac{1}{2}\frac{d^2 f(x^p)}{dx^2}d^2 + R$$

  - for a function with n variables

$$f(\mathbf{x}) = f(\mathbf{x}^p) + \nabla f(\mathbf{x}^p)^T \bullet (\mathbf{x} - \mathbf{x}^p) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^p)^T \bullet \mathbf{H}(\mathbf{x}^p) \bullet (\mathbf{x} - \mathbf{x}^p) + R$$

  - change in the value of the function in moving from x^p to a neighboring point $\mathbf{d} = x - x^p$ distance away from it

$$\Delta f = f(\mathbf{x}^p + \mathbf{d}) - f(\mathbf{x}^p) = \nabla f(\mathbf{x}^p)^T \bullet \mathbf{d} + \frac{1}{2}\mathbf{d}^T \bullet \mathbf{H}(\mathbf{x}^p) \bullet \mathbf{d} + R$$

• Quadratic Forms and Definite Matrices:

- Quadratic form is a special nonlinear function having only second-order terms

For example; $$F(\boldsymbol{x}) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1$$

Representations $$F(\boldsymbol{x}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}x_ix_j = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$$

- Every quadratic form can be put into the following form with a symmetric A matrix

$$F(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \qquad a_{ij} = \frac{1}{2}(p_{ij} + p_{ji})$$

- Many matrices can be associated with a quadratic function, all of them are asymmetric. There is only one unique symmetric matrix.
- The symmetric A matrix determines the nature of the quadratic form.

- For a given value of **x** a quadratic form $F(\mathbf{x}) = 1/2\ \mathbf{x}^T\mathbf{A}\mathbf{x}$ may be either positive, negative, or zero
  - A quadratic form is called positive definite if $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is always positive except for $F(\mathbf{0})$.
  - It is called negative definite if $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$ for all $\mathbf{x}$ except $\mathbf{x} = \mathbf{0}$.
  - If a quadratic form is $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$ and there exists one nonzero $\mathbf{x}$ ($\mathbf{x} \neq \mathbf{0}$) with $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$, then it is called positive semidefinite.
  - If a quadratic form is $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 0$ for all $\mathbf{x}$ and there exists one nonzero $\mathbf{x}$ ($\mathbf{x} \neq \mathbf{0}$) with $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$, then it is called positive semidefinite.
  - A quadratic form which is positive for some vectors **x** and negative for others is called indefinite.

- Check the eigenvalues of the symmetric n×n **A** matrix associated with the quadratic form $F(\mathbf{x}) = 1/2\ \mathbf{x}^T\mathbf{A}\mathbf{x}$.
  - *$F(\mathbf{x})$ is positive definite if and only if all eigenvalues of **A** are strictly positive, i.e. $\lambda_i > 0$, $i = 1$ to n.*
  - *$F(\mathbf{x})$ is positive semidefinite if and only if all eigenvalues of **A** are non-negative, i.e. $\lambda_i \geq 0$, $i = 1$ to n.*
  - *$F(\mathbf{x})$ is negative definite if and only if all eigenvalues of **A** are strictly negative, i.e. $\lambda_i < 0$, $i = 1$ to n.*
  - *$F(\mathbf{x})$ is negative semidefinite if and only if all eigenvalues of **A** are non-positive, i.e. $\lambda_i \leq 0$, $i = 1$ to n.*
  - *$F(\mathbf{x})$ is indefinite if some $\lambda_i < 0$ and some other $\lambda_i > 0$.*